

# Nonabelian Dold-Kan Decompositions for Simplicial and Symmetric-Simplicial Groups

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## Abstract

We extend the nonabelian Dold-Kan decomposition for simplicial groups of Carrasco and Cegarra [CC91] in two ways. First, we show that the total order of the subgroups in their decomposition belongs to a family of total orders all giving rise to Dold-Kan decompositions. We exhibit a particular partial order such that the family is characterized as consisting of all total orders extending the partial order. Second, we consider symmetric-simplicial groups and show that, by using a specially chosen presentation of the category of symmetric-simplicial operators, new Dold-Kan decompositions exist which are algebraically much simpler than those of [CC91] in the sense that the commutator of two component subgroups lies in a single component subgroup.

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# 1 Introduction

In this introduction we refer to the simplicial category **Ord** and the notion of simplicial object in a category  $\mathcal{C}$ , as well as the Moore complex  $M(G)$  of a simplicial group  $G$ . These notions are reviewed in the two subsections of section 2.2.

The classical Dold-Kan theorem ([May67], [GJ99]) says that, in the case in which  $G$  is a simplicial abelian group  $A$ , the terms  $A_n$  of  $A$  decompose as direct sums of copies terms  $M_n = M_n(A)$  of its Moore complex as exemplified below for the cases  $n = 0, 1, 2, 3$ .

$$A_0 \cong M_0$$

$$A_1 \cong M_1 \oplus s_0 M_0$$

$$A_2 \cong M_2 \oplus s_0 M_1 \oplus s_1 M_1 \oplus s_1 s_0 M_0$$

$$A_3 \cong M_3 \oplus s_0 M_2 \oplus s_1 M_2 \oplus s_2 M_2 \oplus s_1 s_0 M_1 \oplus s_2 s_0 M_1 \oplus s_2 s_1 M_1 \oplus s_2 s_1 s_0 M_0$$

For general  $n \geq 0$  one has the isomorphism ([GJ99])

$$A_n \cong M_n \oplus \left( \bigoplus_{k=0}^{n-1} \bigoplus_{\substack{\sigma \in \mathbf{Ord} \\ \sigma: [n] \twoheadrightarrow [k]}} \sigma^* M_k \right)$$

which says that the summands of  $A_n$  are precisely  $M_n$  together with the images in  $A_n$  of lower terms of the Moore complex under all possible degeneracy operators  $\sigma^*$  landing in  $A_n$ . Using the formal structure of this formula, the terms of  $A$  and the action of the simplicial operators on  $A$  can be completely reconstructed from the data contained in  $M(A)$ .

In this way, the Dold-Kan theorem shows that the functor  $M$  is an equivalence of categories between the category **SAb** of simplicial abelian groups and the category **Ch**<sub>+</sub>( $\mathbb{Z}$ ) of nonnegative chain complexes of abelian groups. Under this equivalence, referred to as the *Dold-Kan correspondence*, homomorphic homotopy equivalences of simplicial abelian groups correspond to quasi-isomorphisms of chain complexes. In this sense the Dold-Kan correspondence may be said, on the one hand, to elevate the homological algebra of **Ch**<sub>+</sub>( $\mathbb{Z}$ ) to the level of homotopy theory, and on the other hand, to reduce the homotopy theory of **SAb** to homological algebra.

From this perspective, the general nonabelian case is especially interesting because of the classical result that the category of simplicial groups pos-

sesses a homotopy theory equivalent to that of pointed, connected topological spaces (see [Kan58] and [Qui67]). It would then seem that a nonabelian Dold-Kan theorem could provide a way of translating the homotopy theory of spaces—a subject known for the difficulty in calculating its basic invariants—to a context resembling homological algebra, in which many computational techniques have been developed and standardized. Indeed such a theorem was provided by P. Carrasco and A. M. Cegarra, and they demonstrated its bridging role by using it to describe algebraic models for homotopy types called *hypercrossed complexes* and giving explicit descriptions of truncated hypercrossed complexes yielding models for homotopy 3-types (see [CC91]).

In order to do this, Carrasco and Cegarra first identified the appropriate notion of *higher semidirect product (SDP)* which takes the place of the direct sums in the classical Dold-Kan decomposition (see Definition 3.1 below or the paper [Ant10b] for a more in-depth investigation of SDPs). With this notion in hand, they prove that  $G_n$  is an SDP of copies of terms  $M_n := M_n(G)$  of its Moore complex, as shown here for the cases  $n = 0, 1, 2, 3$ .

$$G_0 = M_0$$

$$G_1 = M_1 \rtimes s_0 M_0$$

$$G_2 = M_2 \rtimes s_0 M_1 \rtimes s_1 M_1 \rtimes s_1 s_0 M_0$$

$$G_3 = M_3 \rtimes s_0 M_2 \rtimes s_1 M_2 \rtimes s_1 s_0 M_1 \rtimes s_2 M_2 \rtimes s_2 s_0 M_1 \rtimes s_2 s_1 M_1 \rtimes s_2 s_1 s_0 M_0$$

One feature of SDPs is that the order of the factors in the decomposition is an essential part of the structure. Therefore an important aspect of the result of [CC91] is a total ordering of the subgroups  $s_{i_k} \dots s_{i_1} M_{n-k} \subseteq G_n$  giving rise to the SDP decomposition of  $G_n$ .

The first goal of the present paper, accomplished in sections 3 and 4, is to extend Carrasco and Cegarra’s nonabelian Dold-Kan decomposition in the following way. Under a particular choice of convention for the Moore complex, their total order takes the form exemplified above, so that we are justified in referring to it as the *binary order*. We exhibit a special partial order on the same collection of subgroups, the *length-product partial order*, such that any total order respecting the length-product partial order will also yield an SDP decomposition of  $G_n$ .

The data necessary to describe hypercrossed complexes, consisting essentially of components of commutator brackets, is admittedly rather complicated. Since it follows from results of Dwyer-Hopkins-Kan (see [DHK85]) and the author (to appear elsewhere) that the category of *symmetric-simplicial*

*groups* (for a definition see paragraph just after Remark 2.1) also possesses a homotopy theory equivalent to that of pointed connected topological spaces, one might wonder whether their extra structure might give rise to hypercrossed complexes of a more manageable character.

We take up this question in section 6, relying on a particular presentation of the category of symmetric-simplicial operators (see Theorem 2.5) derived in [Ant10a] especially for this purpose. The answer is that, if  $G$  is a symmetric-simplicial group, there are new Dold-Kan-type decompositions available for it in addition to the ones described in section 3. For these new decompositions, many more orderings of the components are available than in the earlier case. In effect, the length-product partial order is replaced by the partial order given by inclusion (of sets of indices), and requiring a total order to extend this partial order places many fewer constraints on it. This extra flexibility is a reflection of the fact that the commutators coming from pairs of the component subgroups thus obtained have only a single nontrivial component, giving the data constituting *symmetric hypercrossed complexes* a sleeker form than that of the original hypercrossed complexes. This will be demonstrated in a forthcoming publication.

## 2 Preliminaries

### 2.1 The Simplicial and Symmetric-Simplicial Categories

Recall that a simplicial object in a category  $\mathcal{C}$  is defined as a functor  $\mathbf{Ord}^{op} \rightarrow \mathcal{C}$  where  $\mathbf{Ord}$  is the category of finite ordered sets  $[n] := \{0, 1, \dots, n\}$  and (not necessarily strictly) monotonic maps between them.  $\mathbf{Ord}$  is generated by the *cofaces*  $d_i$  and *codegeneracies*  $s_i$  which are defined by saying that  $d_i : [n] \rightarrow [n+1]$  is the unique monotonic injection for which every fiber (i.e., preimage of a singleton) has one element except for the fiber of  $i$  which has none, while  $s_i : [n] \rightarrow [n-1]$  is the unique monotonic surjection for which every fiber has one element except for the fiber of  $i$  which has two. Here we recall the well-known presentation of the category  $\mathbf{Ord}$  via generators and relations (see [ML70] for an elegant proof). Since in this paper we will follow the tradition of applying simplicial operators on the *left* of simplicial objects, we state the relations in opposite form, i.e., as a presentation of  $\mathbf{Ord}^{op}$ .

**The Simplicial Identities.**

$$\begin{aligned}
d_i d_j &= \begin{cases} d_{j-1} d_i & \text{if } i < j \\ d_j d_{i+1} & \text{if } i \geq j \end{cases} \\
s_i s_j &= \begin{cases} s_{j+1} s_i & \text{if } i \leq j \\ s_j s_{i-1} & \text{if } i > j \end{cases} \\
d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } j + 1 \\ s_j d_{i-1} & \text{if } i \geq j + 2 \end{cases} \quad s_i d_j = \begin{cases} d_{j+1} s_i & \text{if } i < j \\ d_j s_{i+1} & \text{if } i \geq j \end{cases}
\end{aligned}$$

**Remark 2.1** These identities are usually written in a nonredundant form. Here, and in all other presentations below, we have included all possible situations that arise when interchanging two generators, thus incurring a certain amount of redundancy. In particular, the table above (as well as the others to come) is arranged so that all identities in the right column follow from the identities to their left, so that all identities in the right column are redundant. Nevertheless, some redundancies remain in the left column.

Similarly as above, a *symmetric-simplicial object* in a category  $\mathcal{C}$  is defined as a functor  $\mathbf{Fin}^{op} \rightarrow \mathcal{C}$  where  $\mathbf{Fin}$  is the category of finite ordered sets  $[n] := \{0, 1, \dots, n\}$  and *all* maps between them.  $\mathbf{Fin}$  is generated by its subcategory  $\mathbf{Ord}$  together with the groups  $\mathbf{Sym}_n$  for all  $n \geq 0$  of all permutations of the set  $[n]$  (so  $\mathbf{Sym}_n$  is a symmetric group on  $n + 1$  elements). A presentation of  $\mathbf{Fin}$  has been given by Marco Grandis (see [Gra01]) using the generators of  $\mathbf{Ord}$  together with the transpositions  $t_i \in \mathbf{Sym}_n$  (see Definition 2.2 below). Here we shall need an alternative presentation of  $\mathbf{Fin}$  derived in [Ant10a], which uses the generators  $d_i$  and  $t_i$  but replaces the codegeneracies  $s_i$  with the *quasi-codegeneracies*  $u_i$  (see Definition 2.4 below). In order to state the alternative presentation, we first define the relevant maps explicitly.

**Definition 2.2** The following maps in  $\mathbf{Fin}$  are called the *adjacent transpositions* and are defined as follows.

$$\begin{aligned}
t_i &= t_i^{(n)} : [n] \longrightarrow [n] \quad \text{for } n \geq 1 \text{ and } 0 \leq i \leq n - 1 \\
k &\mapsto \begin{cases} k & \text{for } k \neq i, i + 1 \\ i + 1 & \text{for } k = i \\ i & \text{for } k = i + 1 \end{cases}
\end{aligned}$$

**Definition 2.3** The following maps in **Fin** are called the *standard cyclic permutations*.

$$z_i = z_i^{(n)} : [n] \longrightarrow [n] \text{ for } n \geq 1 \text{ and } 0 \leq i \leq n$$

$$k \mapsto \begin{cases} k+1 & \text{for } 0 \leq k \leq i-1 \\ 0 & \text{for } k = i \\ k & \text{for } k > i \end{cases}$$

Note that  $z_i$  is an  $(i+1)$ -cycle on the elements  $0, 1, \dots, i$ . In particular,  $z_0$  is the identity. One may equivalently take the following formula in **Fin**<sup>op</sup> as a definition of the corresponding symmetric-simplicial operator  $z_i$  for  $n \geq 0$  and  $0 \leq i \leq n$ .

$$z_i = z_i^{(n)} := t_{i-1} \dots t_1 t_0$$

**Definition 2.4** The following maps in **Fin** will be referred to as the *elementary quasi-codegeneracies*.

$$u_i = u_i^{(n)} : [n+1] \longrightarrow [n] \text{ for } n \geq 0 \text{ and } 1 \leq i \leq n+1$$

$$k \mapsto \begin{cases} 0 & \text{for } k = 0 \text{ or } i \\ k & \text{for } 1 \leq k \leq i-1 \\ k-1 & \text{for } k > i \end{cases}$$

In particular,  $u_1$  coincides with  $s_0$ . Note  $u_0$  is not defined. One may equivalently define the *elementary quasi-degeneracy operators*  $u_i \in \mathbf{Fin}^{op}$  in terms of the  $s_i$  and  $z_i$  by means of the following formula holding in **Fin**<sup>op</sup> for  $i \geq 1$ .

$$u_i := z_{i-1}^{-1} s_{i-1} z_{i-1}$$

Just as for simplicial objects above, we will also apply symmetric-simplicial operators on the *left* of symmetric-simplicial objects, so we state the alternative presentation of **Fin** in opposite form, i.e., as a presentation of **Fin**<sup>op</sup>.

**Theorem 2.5** *The generators  $d_i$ ,  $u_i$ , and  $t_i$  together with the following relations constitute a presentation of  $\mathbf{Fin}^{op}$ .*

$$\begin{aligned}
d_i d_j &= \begin{cases} d_{j-1} d_i & \text{if } i < j \\ d_j d_{i+1} & \text{if } i \geq j \end{cases} \\
d_i u_j &= \begin{cases} z_{j-1} & \text{if } i = 0 \\ u_{j-1} d_i & \text{if } 0 \neq i < j \\ \text{id} & \text{if } i = j \\ u_j d_{i-1} & \text{if } i > j \end{cases} & u_i d_j = \begin{cases} d_{j+1} u_i & \text{if } i \leq j \\ d_j u_{i+1} & \text{if } i \geq j \neq 0 \\ d_1 u_{i+1} t_0 & \text{if } j = 0 \end{cases} \\
u_i u_j &= \begin{cases} u_{j+1} u_i & \text{if } i \leq j \\ u_j u_{i-1} & \text{if } i > j \end{cases} \\
t_i t_j &= \begin{cases} \text{id} & \text{if } i = j \\ t_j t_i & \text{if } |i - j| \geq 2 \\ (t_j t_i)^2 & \text{if } |i - j| = 1 \end{cases} \\
d_i t_j &= \begin{cases} t_{j-1} d_i & \text{if } i < j \\ d_{i+1} & \text{if } i = j \\ d_{i-1} & \text{if } i = j + 1 \\ t_j d_i & \text{if } i \geq j + 2 \end{cases} & t_i d_j = \begin{cases} d_j t_i & \text{if } i \leq j - 2 \\ d_j t_{i+1} t_i t_{i+1} & \text{if } i = j - 1 \\ d_j t_{i+1} & \text{if } i \geq j \end{cases} \\
t_i u_j &= \begin{cases} u_j t_i & \text{if } 0 \neq i \leq j - 2 \\ u_{j-1} & \text{if } 0 \neq i = j - 1 \\ u_{j+1} & \text{if } i = j \\ u_j t_{i-1} & \text{if } i > j \end{cases} & u_i t_j = \begin{cases} t_{j+1} u_i & \text{if } i \leq j \\ t_j t_{j+1} u_{i-1} & \text{if } i = j + 1 \text{ and } j \neq 0 \\ t_j u_i & \text{if } i \geq j + 2 \text{ and } j \neq 0 \end{cases} \\
t_0 u_1 &= u_1 \\
t_0 u_i t_0 u_j &= \begin{cases} u_{j+1} t_0 u_i t_0 & \text{if } 2 \leq i \leq j \\ u_j t_0 u_{i-1} t_0 & \text{if } 2 \leq j < i \end{cases}
\end{aligned}$$

See [Ant10a] for the proof as well as a discussion of the advantages and disadvantages over Grandis's presentation.  $\blacklozenge$

**Remark 2.6** Since, as mentioned in an earlier remark, all relations in the right column follow from the relations in the left column, all subsequent references to the statement of Theorem 2.5 will be understood as referring to relations of the left column only.

Here are some other useful operators in  $\mathbf{Fin}^{op}$ .

**Definition 2.7** In the statement of Theorem 2.5, note the overlapping conditions in the identities for  $u_i d_j$ . Indeed the equations

$$u_i d_i = d_{i+1} u_i = d_i u_{i+1} =: r_i$$

hold for all  $1 \leq i \leq n$ . We refer to the  $r_i$  as *replacement operators*. They may also be defined directly as functions in  $\mathbf{Fin}$  as follows.

$$r_i = r_i^{(n)} : [n] \longrightarrow [n] \text{ for } n \geq 1 \text{ and } 1 \leq i \leq n$$

$$r_i(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } i \\ k & \text{otherwise} \end{cases}$$

**Proposition 2.8** For each  $n \geq 1$ , the replacement operators

$$r_i : [n] \longrightarrow [n] \text{ for } 1 \leq i \leq n$$

constitute a family of mutually commuting idempotents in  $\mathbf{Fin}^{op}$ .

$$r_i^2 = r_i$$

$$r_i r_j = r_j r_i$$

**Proof.** This is most easily verified using the formula for  $r_i$  as a function in  $\mathbf{Fin}$  given in Definition 2.7. Alternatively, it is a fun exercise to prove the assertion using the algebraic identities of Theorem 2.5.  $\blacklozenge$

## 2.2 Moore Complexes of Simplicial and Symmetric-Simplicial Groups

In this section we briefly recall the construction, due to John C. Moore [Moo55], that takes a simplicial group  $G$  and produces from it its *Moore complex*  $M(G)$ . In this section we also describe extra structure on the Moore complex of a symmetric-simplicial group in the form of actions of permutation operators in  $\mathbf{Fin}^{op}$ .



**Definition 2.9** By the *Moore complex*  $M = M(G)$  of the simplicial group  $G$  we shall mean the chain complex consisting of the nonabelian groups

$$M_n = M_n(G) := \left\{ g \in G_n \mid d_i(g) = 0 \text{ for } i \neq 0 \right\}$$

and boundary operators

$$d := d_0 : M_n \longrightarrow M_{n-1}.$$

The group  $M_n(G)$  will be called the group of *Moore  $n$ -chains* in  $G$ . Also for each  $n \geq 0$  the group of *Moore  $n$ -cycles* is defined as follows.

$$\begin{aligned} Z_0 &= Z_0(G) := M_0(G) \\ Z_n &= Z_n(G) := \bigcap_{0 \leq i \leq n} \ker (d_i : G_n \rightarrow G_{n-1}) \text{ for } n \geq 1 \end{aligned}$$

Finally for each  $n \geq 0$  the group of *Moore  $n$ -boundaries* is defined as follows.

$$B_n = B_n(G) := \text{Image}(d_0 : M_{n+1} \longrightarrow G_n)$$

**Remark 2.10** Expositions about Moore complexes are to be found in [May67] and [GJ99]. Here we cite the following facts for the reader's edification.

- $M(G)$  is a chain complex in the sense that  $d_0^2 : [n+2] \rightarrow [n]$  is the trivial homomorphism for all  $n \geq 0$ .
- The group  $B_n$  is normal in  $Z_n$  for all  $n \geq 0$ .
- The homology groups  $Z_n/B_n$  of  $M(G)$  are naturally isomorphic to the simplicial homotopy groups of  $G$ .

We turn to consider the notion of Moore complex for symmetric-simplicial groups. Considering  $G$  as a simplicial group (i.e., by restricting the action of  $\mathbf{Fin}^{op}$  to its subcategory  $\mathbf{Ord}^{op}$ ), one has the groups  $M_n(G)$  and  $Z_n(G)$  as defined above. Additionally, for all  $n \geq 0$ ,  $G_n$  admits an action of  $\mathbf{Sym}_n$ , the symmetric group on the set  $[n] = \{0, 1, \dots, n\}$ . Let

$$\mathbf{Sym}'_n$$

denote the subgroup of  $\mathbf{Sym}_n$  consisting of the permutations fixing 0.

**Proposition 2.11** *Let  $G$  be a symmetric-simplicial group. The following hold with regard to the action of  $\mathbf{Sym}_n$  on  $G_n$ .*

1. *The subgroup  $M_n(G)$  is invariant under the action of  $\mathbf{Sym}'_n$ .*
2. *The subgroup  $Z_n(G)$  and therefore also  $B_n(G)$  is invariant under the action of  $\mathbf{Sym}_n$ .*

**Proof.** For the first assertion, let  $g$  belong to  $M_n(G)$  so that all faces of  $g$  except  $d_0$  are trivial. Since  $\mathbf{Sym}'_n$  is generated by the transpositions  $t_j$  for  $j > 0$ , it suffices to prove the following claim.

$$d_i t_j(g) = 0 \text{ for all } i, j > 0$$

An examination of the symmetric-simplicial identities (section 2.1) reveals that

$$d_i t_j = t_{j'} d_{i'} \text{ or } d_{i'}$$

for some  $i'$  and  $j'$  where  $i' \neq 0$ , and from this the first assertion follows.

The argument for the second assertion is similar. Taking  $g \in Z_n(G)$ , it suffices to prove the following.

$$d_i t_j(g) = 0 \text{ for all } i, j \geq 0$$

This time, for *any*  $i, j$ , one has

$$d_i t_j = t_{j'} d_{i'} \text{ or } d_{i'}$$

for some  $i'$  and  $j'$ , and hence the second assertion follows. ♦

**Definition 2.12** Let  $G$  be a symmetric-simplicial group. By the *Moore complex of  $G$*  we shall mean the Moore complex  $M(G)$  of the underlying simplicial group of  $G$  together with the actions of  $\mathbf{Sym}_n$  on  $Z_n$  and  $\mathbf{Sym}'_n$  on  $M_n$  and for all  $n$ .

The following definition abstracts the properties of symmetric Moore complexes. Although we shall not need it here, we include it in order to highlight the fact that the structures arising in this way are not the same as the notion of symmetric chain complex encountered in the literature.

**Definition 2.13** Define a *symmetric chain complex* to be a chain complex of (nonabelian) groups

$$M_0 \xleftarrow{d} M_1 \xleftarrow{d} M_2 \xleftarrow{\quad} \cdots$$

together with coextensive actions of  $\mathbf{Sym}_n$  on  $Z_n := \ker(d : M_n \rightarrow M_{n-1})$  and  $\mathbf{Sym}'_n$  on  $M_n$  for all  $n$ . These data are required to satisfy the following condition, which makes use of the generators  $t_0, \dots, t_{n-1}$  of  $\mathbf{Sym}_n$  and in which  $\mathbf{Sym}'_n$  is identified with the subgroup generated by  $t_1, \dots, t_{n-1}$ .

- For all  $m \in M_n$  and all  $i$  with  $1 \leq i \leq n-1$ ,  $d(t_i m) = t_{i-1}(dm)$ .

**Remark 2.14** The above condition is derived from the symmetric-simplicial identity

$$d_0 t_i = t_{i-1} d_0$$

holding for  $i \geq 1$ . For  $i = 0$  one has the identity

$$d_0 t_0 = d_1$$

which corresponds to the fact that in a symmetric chain complex, one always has  $d(t_0 m) = 0$  for  $m \in Z_n$  because  $t_0$  preserves  $Z_n$ .

### 3 Dold-Kan Decompositions for Simplicial Groups

In this section, we generalize a result of [CC91] to the effect that the  $n$ th term  $G_n$  of the simplicial group  $G$  is an internal  $2^n$ -SDP of certain of its subgroups, each isomorphic to some term  $M_j$  of the Moore complex  $M(G)$ . These subgroups are either the normal subgroup  $M_n \triangleleft G_n$  or a copy of  $M_{n'}$  for  $n' < n$  embedded in  $G_n$  as a subgroup of degenerate simplices via an iterated degeneracy operation as follows.

$$M_{n-k} \subseteq G_{n-k} \xrightarrow{s_{i_k} s_{i_{k-1}} \cdots s_{i_1}} G_n$$

For  $n \geq 3$ , there is more than one order in which these subgroups can be arranged to yield an SDP decomposition of  $G_n$ , and we give a characterization of a large family of such orders, which includes (up to choice of convention for the Moore complex) the total order discovered in [CC91]. Although we

believe all such orders are characterized in this fashion, we leave this question open (see Remark 3.4).

Assume given a group  $G$  and subgroups  $H_1, \dots, H_r$ .

**Definition 3.1** The group  $G$  is said to be an *internal  $r$ -semidirect product* (briefly an *SDP*) *of the subgroups  $H_i$*  if the following two conditions hold.

1. The set  $H_1 H_2 \dots H_i$  is a normal subgroup of  $G$  for all  $i$ .
2. Every  $g \in G$  can be factored uniquely as a product

$$g = h_1 h_2 \dots h_r$$

with  $h_i \in H_i$  for all  $i$ .

We follow [CC91] in using the notation

$$G = H_1 \rtimes \dots \rtimes H_r$$

if the above conditions hold.

**Remark 3.2** The order in which the subgroups  $H_i$  appear is an essential part of the definition of SDP. It can happen that  $G$  is an SDP of the  $H_i$  when they are arranged in certain orders, but not in others. At one extreme, there may be a unique such order. At the other extreme, it is immediate that  $G$  is a direct product of the  $H_i$  if and only if  $G$  is an SDP of the  $H_i$  arranged in each possible order.

**Remark 3.3** See [Ant10b] for a proof of the equivalence of Definition 3.1 with the original definition given in [CC91], as well as a further discussion of the properties of semidirect products and a corresponding higher external semidirect product construction.

Let  $G$  be a fixed simplicial group. Following [CC91], *additive notation is used for the group law of each term  $G_n$ , although these groups are generally nonabelian*. Recall that  $G$  possesses a Moore complex  $M(G)$  whose terms will be denoted

$$M_n = M_n(G).$$

The following language and notations will be used. A *multi-index*  $\alpha$  of length  $k$  and dimension  $\leq n$  is a strictly increasing sequence of indices

$$\alpha = \{i_1 < \dots < i_k\}$$

satisfying  $0 \leq i_p \leq n-1$  for all  $p$ . The *length* of  $\alpha$  is the number  $k$  of indices and is denoted by  $|\alpha| := k$ . The set of all multi-indices of dimension  $\leq n$  is denoted as follows.

$$I^{(n)} := \{ \text{multi-indices } \alpha \text{ of dimension } \leq n \}$$

The *degeneracy operator*  $s_\alpha$  corresponding to  $\alpha$  is the composition of elementary degeneracy operators

$$s_\alpha : G_{n-|\alpha|} \longrightarrow G_n$$

$$s_\alpha := s_{i_k} s_{i_{k-1}} \dots s_{i_1}$$

and there is a corresponding subgroup of degenerate simplices

$$H_\alpha := s_\alpha(M_{n-|\alpha|}) \subseteq G_n$$

for each  $|\alpha| \geq 1$ . Note this subgroup is an embedded copy of  $M_{n-|\alpha|}$ , since the degeneracy operator  $s_\alpha$  is injective on account of the simplicial identities  $d_i s_i = d_{i+1} s_i = \text{id}$ . For  $|\alpha| = 0$  there is only the *empty multi-index*  $\alpha = \emptyset$  and the corresponding subgroup

$$H_\emptyset := M_n \triangleleft G_n$$

and its simplices (except the identity element) are all seen to be nondegenerate by noting that any nontrivial simplex of  $M_n$  has at most one face not equal to the identity, whereas a nontrivial degenerate simplex always has at least two equal nontrivial faces ( $y = s_i x$  has the two equal faces  $d_i y = d_{i+1} y = x$ ).

Let us consider the question of how  $G_n$  may decompose as an internal SDP of the groups  $H_\alpha$ , that is, in which ways the  $H_\alpha$  may be totally ordered

yielding such a decomposition. For example, if  $G_n$  is abelian, the  $H_\alpha$  may be arranged in any order. More generally, under various special conditions on the simplicial group  $G$ , there may be especial flexibility in ordering the  $H_\alpha$  to obtain SDP decompositions of  $G_n$ . Here is a more specific question: which total orders on the  $H_\alpha$  give rise to SDP decompositions of  $G_n$  for all simplicial groups  $G$ , regardless of any special characteristics of  $G$ ? Such total orders will here be called *Dold-Kan total orders*, and the corresponding universal decompositions *Dold-Kan decompositions*.

Here we give an answer to this question as follows. There is a partial order on the set  $I^{(n)}$  of multi-indices of dimension  $\leq n$ , here called the *length-product partial order* for lack of a better name, such that any total order extending the length-product partial order will be a Dold-Kan total order. The proof is given in this and the next section, and three examples of Dold-Kan total orders, including the original one appearing in [CC91], are given in the section after that.

**Remark 3.4** The question remains whether this includes *all* Dold-Kan total orders, that is, whether the Dold-Kan total orders are indeed characterized as those extending the length-product partial order. Although we believe the answer to this question is affirmative, we leave it open.

**Definition 3.5** For two multi-indices

$$\alpha = \{i_1 < \dots < i_k\} \quad \beta = \{j_1 < \dots < j_l\},$$

both of degree  $n$ , define the *length-product* partial order relation

$$\alpha \leq \beta$$

to mean first of all that the condition  $k \leq l$  holds, that is,

$$0. \quad |\alpha| \leq |\beta|,$$

and then that the following  $k$  conditions are also satisfied:

1.  $i_1 \leq j_{l-k+1}$ ;
2.  $i_2 \leq j_{l-k+2}$ ;
- $\vdots$
- $k. \quad i_k \leq j_l$

or equivalently that

$$i_{k-p} \leq j_{l-p} \text{ for all } 0 \leq p \leq k-1.$$

The notation  $\alpha < \beta$  indicates the conjunction of  $\alpha \leq \beta$  and  $\alpha \neq \beta$ .

Note the empty multi-index  $\emptyset$  is the absolute minimum of  $I^{(n)}$  under the length-product partial order, and the full multi-index  $\{0 < 1 < \dots < n-1\}$  is the absolute maximum of  $I^{(n)}$ .

In the rest of this section, unless otherwise stated,  $\alpha$  will denote a fixed total order on the multi-indices of dimension  $\leq n$ , thought of as a bijection as follows.

$$\begin{aligned} \alpha(\cdot) : \{0, 1, \dots, 2^n - 1\} &\longrightarrow I^{(n)} \\ k &\mapsto \alpha(k) \end{aligned}$$

On  $\alpha$  the sole requirement is made that it extend the length-product partial order. This is equivalent to the requirement that  $\alpha$  be *order-reflecting*, that is, the following implication holds generally.

$$\alpha(k) \leq \alpha(l) \implies k \leq l$$

In particular,  $\alpha(0)$  is the empty multi-index  $\emptyset$  and  $\alpha(2^n - 1)$  is the full multi-index.

The following theorem is the goal of the section. It is convenient to use the notation  $H_k := H_{\alpha(k)}$ .

**Theorem 3.6** *There is an internal SDP decomposition as follows.*

$$G_n = H_0 \rtimes H_1 \rtimes \dots \rtimes H_{2^n-1}$$

The proof appears at the end of this section.

In the meantime, here are some tools for use in the proof. For an arbitrary multi-index  $\alpha$ , the *face operator* corresponding in a dual manner to  $s_\alpha$  is

$$\begin{aligned} d_\alpha^+ : G_n &\longrightarrow G_{n-|\alpha|} \\ d_\alpha^+ &:= d_{i_1}^+ d_{i_2}^+ \dots d_{i_k}^+ \end{aligned}$$

where the following convenient notation is used.

$$d_i^+ := d_{i+1}$$

There is also a corresponding *projection operator* defined as follows.

$$\begin{aligned}\pi_\alpha : G_n &\longrightarrow G_n \\ \pi_\alpha(g) &:= s_\alpha d_\alpha^+(g)\end{aligned}$$

Similarly as in the statement of Theorem 3.6, the notation  $\pi_k := \pi_{\alpha(k)}$  is used. Note that, by repeated application of the simplicial identities  $d_{i+1}s_i = \text{id}$ , one obtains the identity

$$d_\alpha^+ s_\alpha = \text{id} \tag{1}$$

and consequently  $\pi_k$  is indeed a projection in the sense that the following holds.

$$\pi_k^2 = \pi_k$$

Here is an algorithm that will be shown to produce, for an arbitrary element  $g \in G_n$ , a decomposition of the kind indicated in Theorem 3.6. Define a sequence of elements  $g_i \in G_n$ , starting with  $g_{2^n-1} := g$  and proceeding recursively in reverse order by the formula

$$g_{k-1} := g_k - \pi_k(g_k) \tag{2}$$

and ending with  $g_0$ . By induction one has

$$\begin{aligned}g_k &= g_{k-1} + \pi_k(g_k) \\ &= g_{k-2} + \pi_{k-1}(g_{k-1}) + \pi_k(g_k) \\ &\vdots \\ &= g_0 + \pi_1(g_1) + \pi_2(g_2) + \dots + \pi_k(g_k)\end{aligned}$$

which, following [CC91], we also denote

$$g_k = \sum_{i=0}^k \pi_i(g_i)$$

keeping in mind that the addition here is not commutative in general, so that it is essential to order the summands with indices increasing from left to right. Thus one has the following decomposition for any  $g \in G_n$ .

$$g = g_{2^n-1} = \sum_{i=0}^{2^n-1} \pi_i(g_i) \tag{3}$$



Equation (3) will be referred to as the  $\alpha$ -decomposition of  $g$ .

The proof of the SDP decomposition of Theorem 3.6 above requires certain generalized versions of the simplicial identities as well as an understanding of how the length-product partial order arises from them. This is accomplished by the following theorem, which is proved in detail in the next section.

**Theorem 3.7** *The following facts hold for arbitrary multi-indices*

$$\alpha = \{i_1 < \dots < i_k\} \quad \beta = \{j_1 < \dots < j_l\}$$

*of dimension  $\leq n$ .*

1. *If  $d_\alpha^+ s_\beta = s_{\beta'}$  for some multi-index  $\beta'$ , then  $\alpha \leq \beta$ .*
2. *If  $\alpha \not\leq \beta$  then  $d_\alpha^+ s_\beta = d_{\alpha'}^+ s_{\beta'} d_i$  for some  $\alpha', \beta'$  and some  $i \neq 0$ .*
3. *For  $i \neq 0$ , one has  $d_i d_\beta^+ = d_{\beta'}^+$  for some multi-index  $\beta' > \beta$ .*

The following lemmas refer to the  $\alpha$ -decomposition of an element  $g \in G$  given by (2), (3).

**Lemma 3.8** *The following holds for any  $k$  with  $0 \leq k \leq 2^n - 1$ .*

$$g_k \in \bigcap_{i=k+1}^{2^n-1} \ker d_{\alpha(i)}^+ = \bigcap_{i=k+1}^{2^n-1} \ker \pi_i$$

**Proof.** Proceed by induction (in reverse order). To start, note that the case  $k = 2^n - 1$  is vacuously true.

Now assume the statement holds for some  $k$  with  $0 < k \leq 2^n - 1$ . One checks as follows for any  $i \geq k$  that  $d_{\alpha(i)}^+(g_{k-1})$  is trivial. For  $i = k$ , one has

$$\begin{aligned} d_{\alpha(k)}^+(g_{k-1}) &= d_{\alpha(k)}^+(g_k) - d_{\alpha(k)}^+(\pi_k(g_k)) && \text{(By (2))} \\ &= d_{\alpha(k)}^+(g_k) - d_{\alpha(k)}^+(s_{\alpha(k)} d_{\alpha(k)}^+(g_k)) \\ &= d_{\alpha(k)}^+(g_k) - d_{\alpha(k)}^+(g_k) && \text{(By (1))} \\ &= 0. \end{aligned}$$

For  $i > k$  one starts out similarly, obtaining

$$d_{\alpha(i)}^+(g_{k-1}) = d_{\alpha(i)}^+(g_k) - d_{\alpha(i)}^+ \left( s_{\alpha(k)} d_{\alpha(k)}^+(g_k) \right)$$

of which the first term is trivial by the inductive hypothesis. To see that the second term is also trivial, apply part 2 of Theorem 3.7 to find

$$d_{\alpha(i)}^+ s_{\alpha(k)} d_{\alpha(k)}^+ = d_{\alpha'}^+ s_{\beta'} d_i d_{\alpha(k)}^+$$

for some multi-indices  $\beta$  and  $\gamma$  and some index  $i \neq 0$ . Hence by part 3 of Theorem 3.7 one has

$$d_{\alpha(i)}^+ s_{\alpha(k)} d_{\alpha(k)}^+ = d_{\alpha'}^+ s_{\beta'} d_{\alpha(k')}^+$$

where  $\alpha(k') > \alpha(k)$  and hence  $k' > k$ . Therefore, again by the inductive hypothesis, this operator annihilates  $g_k$  as claimed.  $\blacklozenge$

**Lemma 3.9** *The following holds for any  $k$  with  $0 \leq k \leq 2^n - 1$ .*

$$\pi_k(g_k) \in H_k$$

**Proof.** The assertion of the lemma is that

$$s_{\alpha(k)} d_{\alpha(k)}^+(g_k) \in s_{\alpha(k)} M_{n-|\alpha(k)|}$$

or equivalently

$$d_{\alpha(k)}^+(g_k) \in M_{n-|\alpha(k)|}$$

that is, it suffices to show that  $d_i d_{\alpha(k)}^+(g_k)$  trivial for any  $i \neq 0$ . By part 3 of Theorem 3.7, one has

$$d_i d_{\alpha(k)}^+ = d_{\alpha(k')}^+$$

for some  $\alpha(k') > \alpha(k)$  and hence  $k' > k$ , and so the operator  $d_{\alpha(k')}^+$  annihilates  $g_k$  by Lemma 3.8. Hence  $d_i$  annihilates  $d_{\alpha(k)}^+(g_k)$ , as required.  $\blacklozenge$

**Lemma 3.10** *For any  $k' > k$ , the operator  $\pi_{k'}$  annihilates  $H_k$ .*

**Proof.** It suffices to show that  $d_{\alpha(k')}^+$  annihilates  $s_{\alpha(k)}M_{n-k}$ . Since

$$k' > k \implies \alpha(k') \not\leq \alpha(k)$$

one has by part 2 of Theorem 3.7

$$d_{\alpha(k')}^+ s_{\alpha(k)} M_{n-k} = d_{\alpha'}^+ s_{\beta'} d_i M_{n-k}$$

for some multi-indices  $\beta$  and  $\gamma$  and some index  $i \neq 0$ . But

$$d_i M_{n-k} = \{0\}$$

by definition of the group  $M_{n-k}$ , and the claim follows.  $\blacklozenge$

**Proof of Theorem 3.6.** One must verify the two conditions of Definition 3.1. First it is claimed that, for any  $g \in G_n$ , its  $\alpha$ -decomposition (3) is the unique factorization of  $g$  of the form

$$g = h_0 + h_1 + \dots + h_{2^n-1}$$

with  $h_i \in H_i$  for all  $i$ . That the  $\alpha$ -decomposition of  $g$  is such a factorization is just Lemma 3.9. For uniqueness, let  $g \in G_n$  have two factorizations

$$g = \sum_{i=0}^{2^n-1} h_i = \sum_{i=0}^{2^n-1} h'_i$$

in which  $h_i$  and  $h'_i$  belong to  $H_i$  for all  $i$ . By Lemma 3.10, applying the projection  $\pi_{2^n-1}$  to both sides yields

$$h_{2^n-1} = h'_{2^n-1}$$

and cancelling these terms on the right, one applies the next projection  $\pi_{2^n-2}$  to get the next right-most terms equal. Inductively it follows for all  $i$  that

$$h_i = h'_i.$$

To verify the other condition of the definition of internal  $r$ -SDP, namely that  $H_0 + H_1 + \dots + H_k$  is a normal subgroup of  $G$  for each  $k$ , one must prove the following equality, due in its original form to [CC91].

$$H_0 + H_1 + \dots + H_k = \bigcap_{i=k+1}^{2^n-1} \ker \pi_i$$

From the uniqueness statement just proved, one deduces the following for the  $\alpha$ -decomposition of an element  $g \in G$  that belongs to the left-hand side.

$$\begin{aligned} g = h_0 + \dots + h_k &\iff \pi_{k'}(g_{k'}) = 0 \text{ for all } k' > k \\ &\iff g_{k'} = g \text{ for all } k' \geq k \end{aligned}$$

A simple induction argument shows that the last two statements together are equivalent to

$$\pi_{k'}(g) = 0 \text{ for all } k' > k$$

which says that  $g$  belongs to the right-hand side above, as claimed.  $\blacklozenge$

## 4 Simplicial Identities and the Length-Product Partial Order

The purpose of this section is to prove in detail Theorem 3.7, used in the last section. For convenience, we recall one of the simplicial identities in a slightly modified form.

**A Simplicial Identity.** For  $n \geq 0$  and  $0 \leq i, j \leq n$ , the simplicial operator

$$d_i^+ s_j : G_n \longrightarrow G_n$$

can be rewritten as follows.

$$d_i^+ s_j = \begin{cases} s_{j-1} d_i^+ & \text{if } i < j - 1 \\ \text{id} & \text{if } i = j - 1, j \\ s_j d_{i-1}^+ & \text{if } i > j \end{cases}$$

Note the formulas have the same appearance as the corresponding usual identity except for a shift in the conditions, that is,  $i$  has been replaced by  $i + 1$ . The cases corresponding to  $d_0 s_j$  are absent—although of course still true—reflecting the special role of  $d_0$  as boundary operator in our chosen convention of Moore complex.

The following convenient notations will be used.

$$s_j^- := s_{j-1}$$

$$s_{\beta}^{-} := s_{j_l}^{-} \cdots s_{j_1}^{-}$$

The heart of the proof of Theorem 3.7 is given in the following proposition.

**Proposition 4.1** *Let  $\beta = \{j_1 < \dots < j_l\}$  be a multi-index of dimension  $\leq n$  and let  $i$  be an index satisfying  $0 \leq i \leq n-1$ . Consider the following two alternatives, which are mutually exclusive and cover all possibilities for  $\beta$  and  $i$ . (The nonsensical statements  $j_{l+1} - 1 > i$  and  $i > j_0$  are regarded as true for any  $i$ .)*

1. *For some (unique) subscript  $0 \leq q \leq l$ , one has  $j_{q+1} - 1 > i > j_q$ .*
2. *For some (unique) subscript  $1 \leq q \leq l$ , one has  $j_{q+1} - 1 > i$  and  $i = j_q$  or  $j_q - 1$ .*

*In the first case, the identity*

$$d_i^{+} s_{\beta} = s_{j_l}^{-} s_{j_{l-1}}^{-} \cdots s_{j_{q+1}}^{-} s_{j_q} \cdots s_{j_1} d_{i-q}^{+}$$

*holds, and the rightmost factor  $d_{i-q}^{+}$  is different from  $d_0$ . Then we say that  $d_i^{+}$  has “slipped past”  $s_{\beta}$ . In the second case, the identity*

$$d_i^{+} s_{\beta} = s_{j_l}^{-} s_{j_{l-1}}^{-} \cdots s_{j_{q+1}}^{-} s_{j_{q-1}} \cdots s_{j_1}$$

*holds, and we say that  $d_i^{+}$  is “absorbed by”  $s_{\beta}$ .*

The following lemma, whose statement uses the same notations, is necessary for the proof.

**Lemma 4.2** *The following special cases of Proposition 4.1 hold.*

1. *If  $i > j_l$  then  $d_i^{+} s_{\beta} = s_{\beta} d_{i-|\beta|}^{+}$ .*
2. *If  $i < j_1 - 1$  then  $d_i^{+} s_{\beta} = s_{\beta}^{-} d_i^{+}$ .*

*In each case, the face operator at the right (that is,  $d_{i-|\beta|}^{+}$  or  $d_i^{+}$ ) is not  $d_0$ .*

**Proof.** For the first assertion, note that  $i > j_l$  implies on account of the strictly increasing nature of the indices of  $\beta$

$$i - p > j_l - p \geq j_{l-p}$$

for any  $p$  with  $0 \leq p \leq l - 1$ . Hence the simplicial identity above may be applied repeatedly and one obtains thus

$$\begin{aligned}
d_i^+ s_\beta &= d_i^+ s_{j_l} s_{j_{l-1}} \cdots s_{j_1} \\
&= s_{j_l} d_{i-1}^+ s_{j_{l-1}} \cdots s_{j_1} \\
&= s_{j_l} s_{j_{l-1}} d_{i-2}^+ s_{j_{l-2}} \cdots s_{j_1} \\
&\vdots \\
&= s_{j_l} s_{j_{l-1}} \cdots s_{j_1} d_{i-l}^+ \\
&= s_\beta d_{i-|\beta|}^+
\end{aligned}$$

as claimed. To see that  $d_{i-|\beta|}^+$  is not  $d_0$ , note that because the  $j_q$  are strictly increasing one has  $j_q \geq q - 1$  for each  $q$ . From the hypothesis  $i > j_l$  it then follows that  $i \geq |\beta|$  and so indeed  $d_{i-|\beta|}^+$  cannot be  $d_0$ .

For the second assertion, note it follows from the increasing nature of the indices of  $\beta$  and the hypothesis  $i < j_1 - 1$  that

$$i < j_q - 1$$

for all  $q$ . Then, again successively applying the simplicial identity, one obtains

$$\begin{aligned}
d_i^+ s_\beta &= d_i^+ s_{j_l} s_{j_{l-1}} \cdots s_{j_1} \\
&= s_{j_l}^- d_i^+ s_{j_{l-1}} \cdots s_{j_1} \\
&= s_{j_l}^- s_{j_{l-1}}^- d_i^+ s_{j_{l-2}} \cdots s_{j_1} \\
&\vdots \\
&= s_{j_l}^- s_{j_{l-1}}^- \cdots s_{j_1}^- d_i^+ \\
&= s_\beta^- d_i^+
\end{aligned}$$

as claimed. From  $i \geq 0$  it immediately follows that  $d_i^+$  is not  $d_0$ . ◆

**Proof of Proposition 4.1.** If  $\beta$  contains neither  $i$  nor  $i + 1$ , then the situation of part 1 of the Proposition obtains. In that case, there is a factorization

$$s_\beta = s_{\beta'} s_{\beta''}$$

where  $\beta'$  consists of those indices of  $\beta$  greater than  $i + 1$  and  $\beta''$  consists of those indices of  $\beta$  less than  $i$ . Applying Lemma 4.2, one obtains

$$\begin{aligned} d_i^+ s_\beta &= d_i^+ s_{\beta'} s_{\beta''} \\ &= s_{\beta'}^- d_i^+ s_{\beta''} && \text{(By Lemma 4.2, part 1)} \\ &= s_{\beta'}^- s_{\beta''} d_{i-|\beta''|}^+ && \text{(By Lemma 4.2, part 2)} \end{aligned}$$

where  $d_{i-|\beta''|}^+$  is not  $d_0$ . This proves part 1 of the Proposition.

Now assume  $\beta$  contains one or both of  $i$  and  $i + 1$ , so that the situation of part 2 of the Proposition obtains. Letting  $p$  stand for the larger of  $i$  and  $i + 1$  contained in  $\beta$ , there is a factorization

$$s_\beta = s_{\beta'} s_p s_{\beta''}$$

where  $\beta'$  consists of those indices of  $\beta$  greater than  $p + 1$  and  $\beta''$  consists of those indices of  $\beta$  less than  $p$ . Again applying Lemma 4.2, one obtains

$$\begin{aligned} d_i^+ s_\beta &= d_i^+ s_{\beta'} s_p s_{\beta''} \\ &= s_{\beta'}^- d_i^+ s_p s_{\beta''} && \text{(By Lemma 4.2, part 1)} \\ &= s_{\beta'}^- s_{\beta''} && \text{(By the simplicial identity)} \end{aligned}$$

thus proving part 2 of the Proposition.  $\blacklozenge$

We turn to the proof of Theorem 3.7 from the previous section. Recall that there the rank-product partial order  $\leq$  was defined on the multi-indices of dimension  $\leq n$ .

**Theorem 3.7** The following facts hold for arbitrary multi-indices

$$\alpha = \{i_1 < \dots < i_k\} \quad \beta = \{j_1 < \dots < j_l\}$$

of dimension  $\leq n$ .

1. If  $d_\alpha^+ s_\beta = s_{\beta'}$  for some multi-index  $\beta'$ , then  $\alpha \leq \beta$ .
2. If  $\alpha \not\leq \beta$  then  $d_\alpha^+ s_\beta = d_{\alpha'}^+ s_{\beta'} d_i$  for some  $\alpha', \beta'$  and some  $i \neq 0$ .
3. For  $i \neq 0$ , one has  $d_i d_\beta^+ = d_{\beta'}^+$  for some multi-index  $\beta' > \beta$ .

**Proof of Theorem 3.7, part 1.** The strategy is to work with the left hand side of

$$d_{\alpha}^{+} s_{\beta} = s_{\beta'}$$

using the simplicial identities to push the elementary factors of  $d_{\alpha}^{+}$  one at a time across the sequence of factors of  $s_{\beta}$  and to observe in the process that the various conditions constituting the assertion  $\alpha \leq \beta$  hold.

First observe that, according to the dichotomy of Proposition 4.1, as each factor of  $d_{\alpha}^{+}$  is pushed through  $s_{\beta}$ , it will either cancel a factor of  $s_{\beta}$  or it will slip past  $s_{\beta}$ . In the present case, it is not possible for a factor to slip past, because any product of elementary operators starting with a face operator  $d_i$  on the right corresponds to a monotonic function in **Ord** failing to have  $i$  as a value, whereas  $s_{\beta'}$  corresponds to a surjective function in **Ord**. One concludes that each factor of  $d_{\alpha}^{+}$  cancels some factor of  $s_{\beta}$ , and consequently

$$|\alpha| \leq |\beta|.$$

To verify the remaining conditions, now begin by pushing  $d_{i_k}^{+}$  across  $s_{\beta}$ . Under Proposition 4.1, one may say that there is a unique subscript  $q(k)$  with  $0 \leq q(k) \leq l$  satisfying

$$\begin{aligned} j_{q(k)+1} - 1 &> i_k \\ i_k &= j_{q(k)} \quad \text{or} \quad j_{q(k)} - 1 \end{aligned} \tag{4}$$

so that  $d_{i_k}^{+}$  cancels with  $s_{j_{q(k)}}$ , and moreover the result is

$$d_{i_k}^{+} s_{\beta} = s_{j_l}^{-} s_{j_{l-1}}^{-} \cdots s_{j_{q(k)+1}}^{-} s_{j_{q(k)-1}} \cdots s_{j_1} =: s_{\beta(k)}.$$

Now we seek to push the next face operator  $d_{i_{k-1}}^{+}$  across  $s_{\beta(k)}$ . Since  $d_{i_k}^{+}$  slipped past  $s_{j_l}, \dots, s_{j_{q(k)+1}}$ , the next factor  $d_{i_{k-1}}^{+}$  will also slip past  $s_{j_l}^{-}, \dots, s_{j_{q(k)+1}}^{-}$  on account of the strict inequality  $i_{k-1} < i_k$ . To be precise, one has the conditions

$$\begin{aligned} i_{k-1} &< i_k < j_{q(k)+1} - 1 < \cdots < j_l - 1 \\ \implies i_{k-1} &< j_{q(k)+1} - 2 < \cdots < j_l - 2 \end{aligned}$$

enabling us to apply the simplicial identity to get

$$d_{i_{k-1}}^{+} s_{\beta(k)} = s_{j_l}^{--} s_{j_{l-1}}^{--} \cdots s_{j_{q(k)+1}}^{--} d_{i_{k-1}}^{+} s_{j_{q(k)-1}} \cdots s_{j_1}$$



where  $s_j^{--} := s_{j-2}$ . As  $d_{i_{k-1}}^+$  is pushed further to the right, again there must be a unique subscript  $q(k-1)$ , evidently less than  $q(k)$ , such that  $d_{i_{k-1}}^+$  cancels with  $s_{j_{q(k-1)}}$ . Again by Proposition 4.1, the result is

$$\begin{aligned} d_{i_{k-1}}^+ s_{\beta(k)} &= s_{j_l}^{--} s_{j_{l-1}}^{--} \cdots s_{j_{q(k)+1}}^{--} \circ \\ &\quad s_{j_{q(k)-1}}^{--} s_{j_{q(k)-2}}^{--} \cdots s_{j_{q(k-1)+1}}^{--} \circ \\ &\quad s_{j_{q(k-1)-1}} s_{j_{q(k-1)-2}} \cdots s_{j_1} =: s_{\beta(k-1)}. \end{aligned}$$

Continuing in this manner, one obtains a sequence of subscripts  $q(p)$  with

$$l \geq q(k) > q(k-1) > \cdots > q(1) \geq 0 \quad (5)$$

and a sequence of degeneracy operators  $s_{\beta(p)}$  such that

$$d_{i_p}^+ s_{\beta(p+1)} =: s_{\beta(p)}$$

and such that in pushing  $d_{i_p}^+$  across  $s_{\beta(p+1)}$ , it cancels with  $s_{j_{q(p)}}$  and hence does not affect the indices of the operators  $s_{j_q}$  further to the right (that is, the indices  $q < q(p)$ ). Since the sequence of indices  $j_q$  is strictly increasing, one has

$$\begin{aligned} i_p &\leq j_{q(p)} && \text{(By (4))} \\ &= j_{q(k-k+p)} \\ &\leq j_{q(k)-k+p} && \text{(By (5))} \\ &\leq j_{l-k+p} && \text{(since } q(k) \leq l) \end{aligned}$$

for all  $p$  with  $1 \leq p \leq k$ , and thus it is verified that  $\alpha \leq \beta$ .  $\blacklozenge$

**Proof of Theorem 3.7, part 2.** Recall that, by Proposition 4.1, in pushing each factor of  $d_\alpha^+$  across  $s_\beta$ , one of two possibilities can occur: either the factor cancels somewhere along the way, or it makes it all the way through to the right. If cancellation occurred for each factor, then by the just-proved part 1 of the Theorem, one would have  $\alpha \leq \beta$ . Since it was assumed that  $\alpha \not\leq \beta$ , one of the factors must make it through.

Using the notation from the proof of part 1, let us say that  $d_{i_p}^+$  is the first

factor to make it through. Prior to this occurrence, one has

$$\begin{aligned}
d_\alpha^+ s_\beta &= d_{i_1}^+ \dots d_{i_{k-2}}^+ d_{i_{k-1}}^+ d_{i_k}^+ s_\beta \\
&= d_{i_1}^+ \dots d_{i_{k-2}}^+ d_{i_{k-1}}^+ s_{\beta(k)} \\
&= d_{i_1}^+ \dots d_{i_{k-2}}^+ s_{\beta(k-1)} \\
&\vdots \\
&= d_{i_1}^+ \dots d_{i_{p-1}}^+ d_{i_p}^+ s_{\beta(p+1)}
\end{aligned}$$

The hypotheses of Proposition 4.1, part 1 apply to  $d_{i_p}^+ s_{\beta(p+1)}$ , for otherwise there would be cancellation of  $d_{i_p}^+$ . Therefore one gets finally

$$d_\alpha^+ s_\beta = d_{i_1}^+ \dots d_{i_{p-1}}^+ s_{\beta'} d_i^+$$

where  $i \neq 0$ . ◆

**Proof of Theorem 3.7, part 3.** For this final proof, we shed our previous labelling habits and index  $d_\beta^+$  directly, writing

$$d_\beta^+ = d_{j_1} d_{j_2} \dots d_{j_l}$$

$$1 \leq j_1 < \dots < j_l \leq n$$

and consider how the simplicial identities give rise to the multi-index  $\beta'$  in the equation  $d_\beta^+ = d_i d_\beta^+$ . The relevant computation

$$\begin{aligned}
d_i d_\beta^+ &= d_{i+0} d_{j_1} d_{j_2} d_{j_3} \dots d_{j_l} \\
&= d_{j_1} d_{i+1} d_{j_2} d_{j_3} \dots d_{j_l} \\
&= d_{j_1} d_{j_2} d_{i+2} d_{j_3} \dots d_{j_l} \\
&\vdots \\
&= d_{j_1} \dots d_{j_q} d_{i+q} d_{j_{q+1}} \dots d_{j_l}
\end{aligned}$$

is explained as follows. As long as

$$i + m \geq j_{m+1}$$

holds, one may push  $d_{i+m}$  (the avatar of  $d_i$ ) to the right using the simplicial identity

$$d_{i+m} d_{j_{m+1}} = d_{j_{m+1}} d_{i+m+1}$$

until the first subscript  $q$  is reached such that

$$i + q < j_{q+1},$$

at which point the indices are in increasing order from left to right. Since the product on the right-hand-side evidently does not contain  $d_0$  as a factor, there exists a multi-index  $\beta'$  such that the final result  $d_{\beta'}^+$ . Now reindex it directly as

$$d_{\beta'}^+ = d_{j'_1} d_{j'_2} \dots d_{j'_{l+1}}$$

$$j'_p = \begin{cases} j_p & \text{if } 1 \leq p \leq q \\ i + q & \text{if } p = q + 1 \\ j_{p-1} & \text{if } p \geq q + 2. \end{cases}$$

It is straightforward to verify  $j_p \leq j'_{1+p}$  for each  $p$  with  $1 \leq p \leq l$ , showing  $\beta \leq \beta'$ . Due to the fact that

$$|\beta'| = |\beta| + 1$$

it must be that  $\beta \neq \beta'$  and so one concludes  $\beta < \beta'$  as claimed.  $\blacklozenge$

## 5 On the Binary Total Order

In this section, we describe the Dold-Kan total order discovered by Carrasco and Cegarrà [CC91]. The difference in its appearance here is due to our choice of convention for the Moore complex (see section 2.2). Written this way, it is given by binary representations of the natural numbers, and so it seems reasonable to call it the *binary order*.

**Definition 5.1** The *binary total order*  $\alpha(\cdot)$  is defined as follows. For a nonnegative integer  $k$  with binary expansion

$$k = \sum_{i=0}^{n-1} c_i 2^i$$

where  $c_i$  is the  $i$ th binary digit of  $k$ , set

$$\alpha(k) := \{ i \mid c_i = 1 \}.$$

For  $n = 4$  the total order appears as follows. The indices are written as they appear in the subscripts of degeneracy operators, that is, in decreasing order.

$$\begin{array}{ccccccc} \emptyset & < & 0 & < & 1 & < & 10 \\ < & 2 & < & 20 & < & 21 & < & 210 \\ < & 3 & < & 30 & < & 31 & < & 310 \\ < & 32 & < & 320 & < & 321 & < & 3210 \end{array}$$

In general,  $\alpha'$  is strictly less than  $\alpha$  in the binary order if and only if the inequality

$$\sum_{j \in \alpha'} 2^j < \sum_{j \in \alpha} 2^j$$

holds, that is, in reading the corresponding binary expansions from left to right (greatest to least), if the  $p$ -th digit is the first place in which they differ, then the  $p$ -th digit of  $\alpha'$  is a 0 and for  $\alpha$  it is 1 (i.e.,  $p$  belongs to  $\alpha$  but not to  $\alpha'$ ).

**Remark 5.2** A different, but equivalent, description is that  $\alpha'$  is strictly less than  $\alpha$  in the binary order if and only if the corresponding degeneracy operators are of the form

$$\begin{aligned} s_{\alpha'} &= s_{i_k} s_{i_{k-1}} \dots s_{i_{q+1}} s_{i'_q} s_{i'_{q-1}} \dots \\ s_{\alpha} &= s_{i_k} s_{i_{k-1}} \dots s_{i_{q+1}} s_{i_q} s_{i_{q-1}} \dots \end{aligned}$$

where the index  $i'_q$  is either strictly less than  $i_q$  or nonexistent (we assume the labelling here to be such that either  $\alpha'$  or  $\alpha$  may have rank  $k$  or less). That is, comparing the indices of  $\alpha'$  and  $\alpha$  in order from greatest to least, either  $\alpha$  has the greater index in the first position  $q$  in which they differ, or they coincide in each position up until the point that  $\alpha'$  runs out of indices and  $\alpha$  still has some left.

Essentially for this reason, Carrasco and Cegarra [CC91] call this order *lexicographic*, and it also appears this way in the indices of degeneracy operators, that is, when the indices are written in decreasing order, as was done above. According to our chosen conventions, however (namely, our habit of writing multi-indices as increasing from left to right and also our choice of convention for the Moore complex), binary order would be called *reverse-lexicographic*. We use the name *binary* partly in order to avoid confusion on this point.

**Remark 5.3** Carrasco and Cegarra discovered a remarkable explicit formula for the individual components  $\pi_k(g_k)$  (see section 3) holding when binary order is used, and the reader is referred to their paper [CC91] for its proof. The purpose of this remark is to give one way of stating their formula using the present conventions. The notations of the section 3 will be freely used.

Fix an index  $m$  with  $0 \leq m \leq 2^n - 1$ , and let

$$\alpha = \{i_1 < \dots < i_k\}$$

stand for  $\alpha(m)$ . Also write  $\alpha^c$  for the multi-index which is the complement of  $\alpha$  in  $\{0, \dots, n-1\}$  and write

$$\alpha^c = \{j_1 < \dots < j_l\}.$$

Define the homomorphisms

$$q_i : G_n \longrightarrow G_n$$

$$q_i(g) := s_i d_i^+(g)$$

for  $0 \leq i \leq n-1$ , and let

$$q_\alpha := q_{i_k} \dots q_{i_1}$$

be the product of  $q_i$  for  $i \in \alpha$  in *decreasing* order from left to right. Also write  $q_j^\perp$  for the crossed homomorphisms

$$q_j^\perp : G_n \longrightarrow G_n$$

$$q_j^\perp := 1 - q_j$$

$$q_j^\perp(g) = g - s_j d_j^+(g)$$

and write  $q_{\alpha^c}^\perp$  for the product

$$q_{j_1}^\perp \dots q_{j_l}^\perp$$

of  $q_j$  for  $j \in \alpha^c$  in *increasing* order from left to right.

Then the component  $\pi_k(g_k)$  of  $g \in G_n$  under the SDP decomposition of the previous section using binary order can be expressed explicitly in terms of  $g$  by the following formula.

$$\pi_k(g_k) = q_\alpha q_{\alpha^c}^\perp(g)$$

## 6 Alternative Dold-Kan Decompositions for Symmetric Simplicial Groups

In this section, we show that, if  $G$  is a symmetric-simplicial group, there are new Dold-Kan-type decompositions available for it in addition to the ones described in section 3. Once again the  $n$ th term  $G_n$  is a  $2^n$ -fold internal SDP of certain of its subgroups, each isomorphic to some term  $M_j$  of the Moore complex  $M(G)$ . The main difference is that the subgroups  $s_\alpha M_{n-|\alpha|}$  of degenerate simplices are replaced by subgroups  $u_\alpha M_{n-|\alpha|}$  of quasidegenerate simplices.

As to the question of the ordering of these subgroups, many more orderings are available than in the earlier case. In effect, the length-product partial order is replaced by the partial order given by inclusion, and requiring a total order to extend this partial order places many fewer constraints on it.

Throughout this section, assume given a fixed symmetric-simplicial group  $G$  and work with a fixed term  $G_n$  of  $G$ .

The language and notations of section 3 will be reused with one change as follows. A *symmetric multi-index  $\alpha$  of length  $k$  and dimension  $\leq n$*  is a strictly increasing sequence of indices

$$\alpha = \{i_1 < \dots < i_k\}$$

satisfying  $1 \leq i_p \leq n$  for all  $p$  (this is the change—the allowed range of the indices is shifted upwards by 1). The length  $|\alpha|$  of  $\alpha$  is once again the number  $k$  of indices. Denote the set of all symmetric multi-indices of dimension  $\leq n$  as follows.

$$J^{(n)} := \{ \text{symmetric multi-indices } \alpha \text{ of dimension } \leq n \}$$

For brevity, symmetric multi-indices will be called simply multi-indices. Hopefully this will cause no confusion.

The (*generalized*) *quasidegeneracy operator*  $u_\alpha$  corresponding to  $\alpha$  is the composition of elementary quasidegeneracy operators

$$u_\alpha : G_{n-|\alpha|} \longrightarrow G_n$$

$$u_\alpha := u_{i_k} u_{i_{k-1}} \dots u_{i_1}$$

and there is a corresponding subgroup of quasidegenerate simplices

$$U_\alpha := u_\alpha(M_{n-|\alpha|}) \subseteq G_n$$

for  $|\alpha| \geq 1$ . Note this subgroup is an embedded copy of  $M_{n-|\alpha|}$ , since the quasidegeneracy operator  $u_\alpha$  is injective on account of the identity  $d_i u_i = \text{id}$  from Theorem 2.5. For  $\alpha = \emptyset$ , there is once again the corresponding subgroup

$$U_\emptyset := M_n \triangleleft G_n.$$

As in section 3, let us consider the question of which total orderings of the  $U_\alpha$  give rise to SDP decompositions of  $G_n$  for any symmetric-simplicial group  $G$ . Such total orders and their corresponding decompositions will be called *symmetric Dold-Kan*.

Our answer to this question is given in a similar manner. Letting  $J^{(n)}$  be partially ordered by inclusion, any total order of  $J^{(n)}$  extending the inclusion partial order will be a symmetric Dold-Kan total order.

In the rest of this section,  $\alpha(\cdot)$  will denote a fixed total order on the multi-indices of dimension  $\leq n$ , thought of as a bijection as follows.

$$\alpha(\cdot) : \{0, 1, \dots, 2^n - 1\} \longrightarrow J^{(n)}$$

$$k \mapsto \alpha(k)$$

On  $\alpha(\cdot)$  the sole requirement is made that it extend the inclusion partial order, or equivalently

$$\alpha(k) \subseteq \alpha(l) \implies k \leq l.$$

Once again, this forces  $\alpha(0)$  to be the empty multi-index  $\emptyset$  and  $\alpha(2^n - 1)$  to be the full multi-index.

The following theorem is the goal of the section. Write  $U_k := U_{\alpha(k)}$ .

**Theorem 6.1** *There is an internal SDP decomposition as follows.*

$$G_n = U_0 \rtimes U_1 \rtimes \dots \rtimes U_{2^n-1}$$

Here are tools analogous to those used in section 3. For an arbitrary multi-index  $\alpha$ , the *face operator* corresponding in a dual manner to  $u_\alpha$  is

$$\begin{aligned} d_\alpha : G_n &\longrightarrow G_{n-|\alpha|} \\ d_\alpha &:= d_{i_1} d_{i_2} \dots d_{i_k} \end{aligned}$$

There is a corresponding *projection operator* defined as follows.

$$\begin{aligned} \pi_\alpha : G_n &\longrightarrow G_n \\ \pi_\alpha(g) &:= u_\alpha d_\alpha(g) \end{aligned}$$

Similarly as in the statement of Theorem 6.1, we again use the notation  $\pi_k := \pi_{\alpha(k)}$ . By repeated application of the identities  $d_i u_i = \text{id}$  (section 2.1), one obtains

$$\begin{aligned} d_\alpha u_\alpha &= \text{id} \\ \pi_k^2 &= \pi_k. \end{aligned}$$

The decomposition algorithm used here is identical in appearance to the one used in section 3. Namely, for  $g \in G_n$ , define a sequence of elements  $g_i \in G_n$ , starting with  $g_{2^n-1} := g$  and proceed recursively in reverse order by the formula

$$g_{k-1} := g_k - \pi_k(g_k) \tag{6}$$

and ending with  $g_0$ . Thus one has the following decomposition for any  $g \in G_n$ , in which one must once again be careful to keep the summands in increasing order (left to right).

$$g = g_{2^n-1} = \sum_{i=0}^{2^n-1} \pi_i(g_i) \tag{7}$$

The proof of the SDP decomposition of Theorem 6.1 above requires the following “quasi” analogs of Proposition 4.1 and Theorem 3.7. The proofs of these analogs are nearly the same as those of the originals, so we leave the verifications to the interested reader with the advice simply to follow the original proofs closely, accounting carefully for the differences in the statements of Propositions 4.1 and 6.2. and inserting the inclusion partial order in place of the length-product partial order wherever it occurs.

The following convenient notation will be used.

$$u_j^- := u_{j-1}$$



**Proposition 6.2** *Let  $\beta = \{j_1 < \dots < j_l\}$  be a symmetric multi-index of dimension  $\leq n$  and let  $i$  be an index satisfying  $1 \leq i \leq n$ . Consider the following two alternatives, which are mutually exclusive and cover all possibilities for  $\beta$  and  $i$ . (The nonsensical statements  $j_{l+1} > i$  and  $i > j_0$  are agreed to be true for any  $i$ .)*

1. *For some (unique) subscript  $0 \leq q \leq l$ , one has  $j_{q+1} > i > j_q$ .*
2. *For some (unique) subscript  $1 \leq q \leq l$ , one has  $i = j_q$ .*

*In the first case, the identity*

$$d_i u_\beta = u_{j_l}^- u_{j_{l-1}}^- \dots u_{j_{q+1}}^- u_{j_q} \dots u_{j_1} d_{i-q}$$

*holds, and the rightmost factor  $d_{i-q}$  is different from  $d_0$ . Then we say that  $d_i$  has “slipped past”  $u_\beta$ . In the second case, the identity*

$$d_i u_\beta = u_{j_l}^- u_{j_{l-1}}^- \dots u_{j_{q+1}}^- u_{j_{q-1}} \dots u_{j_1}$$

*holds, and we say that  $d_i$  is “absorbed by”  $u_\beta$ .*

**Theorem 6.3** *The following facts hold for arbitrary multi-indices*

$$\alpha = \{i_1 < \dots < i_k\} \quad \beta = \{j_1 < \dots < j_l\}$$

*of dimension  $\leq n$ .*

1.  $d_\alpha u_\beta = u_{\beta'}$  for some multi-index  $\beta'$  if and only if  $\alpha \subseteq \beta$ .
2. If  $\alpha \not\subseteq \beta$  then  $d_\alpha u_\beta = d_{\alpha'} u_{\beta'} d_i$  for some  $\alpha', \beta'$  and some  $i \neq 0$ .
3. For  $i \neq 0$ , one has  $d_i d_\beta = d_{\beta'}$  for some multi-index  $\beta' \supsetneq \beta$ .

The next three lemmas are the analogs of Lemmas 3.8, 3.9 and 3.10. The difference is that the operators  $s_\alpha$  are replaced by  $u_\alpha$  and the role of Theorem 3.7 is taken over by Theorem 6.3. The proofs are otherwise completely identical, and so we omit them.

**Lemma 6.4** *The following holds for any  $k$  with  $0 \leq k \leq 2^n - 1$ .*

$$g_k \in \bigcap_{i=k+1}^{2^n-1} \ker d_{\alpha(i)} = \bigcap_{i=k+1}^{2^n-1} \ker \pi_i$$

**Lemma 6.5** *The following holds for any  $k$  with  $0 \leq k \leq 2^n - 1$ .*

$$\pi_k(g_k) \in U_k$$

**Lemma 6.6** *For any  $k' > k$ , the operator  $\pi_{k'}$  annihilates  $U_k$ .*

The proof of Theorem 6.1 is now practically identical to the proof of Theorem 3.6, and so we omit it as well.

One can define the length-product partial order for  $J^{(n)}$  exactly as was done for  $I^{(n)}$  in section 3. The following proposition shows that the length-product partial order extends the inclusion partial order.

**Proposition 6.7** *For multi-indices  $\alpha$  and  $\beta$  as in Theorem 6.3, the following implication holds.*

$$\alpha \subseteq \beta \implies \alpha \leq \beta$$

**Proof.** First note that

$$\alpha \subseteq \beta \implies |\alpha| \leq |\beta|$$

so that the 0th condition of  $\alpha \leq \beta$  is satisfied. For the rest, define the following function.

$$q(\cdot) : \{1, \dots, k\} \longrightarrow \{1, \dots, l\}$$

$$p \mapsto \text{the number } q(p) \text{ such that } i_p = j_{q(p)}$$

Note  $q(\cdot)$  is then a *strictly* increasing function, so that for each  $p$  one has

$$\begin{aligned} q(k-p) &< q(k-p+1) < \dots < q(k) \leq l \\ \implies q(k-p) &\leq l-p. \end{aligned}$$

Since the  $j_q$  are increasing in  $q$  one has immediately

$$i_{k-p} = j_{q(k-p)} \leq j_{l-p}$$

as required in the definition of  $\alpha \leq \beta$ . ◆

As a result of this proposition, any total order extending the length-product partial order also extends the inclusion partial order. In particular, the binary total order of section 5 can be used as a symmetric Dold-Kan total order. This is done in the final remark below in order to investigate the symmetric analog of the formula from Remark 5.3.

**Remark 6.8** The following remark gives the symmetric analog of the formula of Carrasco and Cegarra from Remark 5.3. Fix an index  $m$  with  $0 \leq m \leq 2^n - 1$ , and let

$$\alpha = \{i_1 < \dots < i_k\}$$

stand for  $\alpha(m)$ . Also write  $\alpha^c$  for the multi-index which is the complement of  $\alpha$  in  $\{1, \dots, n\}$  and write

$$\alpha^c = \{j_1 < \dots < j_l\}.$$

Recall from Definition 2.7 the homomorphisms

$$\begin{aligned} r_i &: G_n \longrightarrow G_n \\ r_i(g) &:= u_i d_i(g) \end{aligned}$$

for  $1 \leq i \leq n$ , and let

$$r_\alpha := r_{i_k} \dots r_{i_1}$$

be the product of  $r_i$  for  $i \in \alpha$  (order does not matter as they commute). Also write  $r_j^\perp$  for the crossed homomorphisms

$$\begin{aligned} r_j^\perp &: G_n \longrightarrow G_n \\ r_j^\perp &:= 1 - r_j \\ r_j^\perp(g) &= g - u_j d_j(g) \end{aligned}$$

and write  $r_{\alpha^c}^\perp$  for the product

$$r_{j_1}^\perp \dots r_{j_l}^\perp$$

of the  $r_j^\perp$  for  $j \in \alpha^c$  in *increasing* order from left to right.

Then the component  $\pi_k(g_k)$  of  $g \in G_n$  under the SDP decomposition of the present section using binary order can be expressed explicitly in terms of  $g$  by the following formula.

$$\pi_k(g_k) = r_\alpha r_{\alpha^c}^\perp(g)$$

Due to the relations

$$r_i(1 - r_j) = (1 - r_j)r_i$$

holding for all  $i, j$ , the factors  $r_i$  may be mixed around among the factors  $1 - r_j$ . In particular, they may be arranged into the product in increasing order

$$\pi_k(g_k) = r_1^\varepsilon r_2^\varepsilon \dots r_n^\varepsilon$$

where  $r_i^\varepsilon$  is  $r_i$  or  $(1 - r_i)$  depending as  $i$  does or does not belong to  $\alpha$ .

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